

ON DOMINANT RATIONAL MAPS FROM PRODUCTS OF CURVES TO SURFACES OF GENERAL TYPE

F. BASTIANELLI AND G. P. PIROLA

ABSTRACT. In this paper we investigate the existence of generically finite dominant rational maps from products of curves to surfaces of general type. We prove that the product $C \times D$ of two distinct very general curves of genus $g \geq 7$ and $g' \geq 2$ does not admit dominant rational maps—other than the identity—on surfaces of general type.

1. INTRODUCTION

Let X be a non-singular complex projective variety of general type. Let $\mathcal{IS}(X)$ be the *Iitaka-Severi set* of X , whose elements are equivalence classes of generically finite dominant rational maps $X \dashrightarrow Y$ to varieties of general type, where two such maps are equivalent if they differ for a birational isomorphism $Y \dashrightarrow Y'$ of the target varieties. Moreover, let us denote by $s(X)$ its cardinality.

The finiteness of $\mathcal{IS}(X)$ has been recently proved by means of the main result of [25] and important advances in the knowledge of pluricanonical maps (cf. [35, 33, 19]). Furthermore, there are several works providing bounds on $s(X)$ (see e.g. [22, 34, 15, 17, 18] and [21, 29] under additional hypothesis).

When X is a curve of genus g with general moduli (i.e. for any point $[X] \in \mathcal{M}_g$ contained in a certain non-empty Zariski open subset of the moduli space) it is well known that X does not dominate other curves of genus greater than one. Equivalently,

Theorem 1.1. *Let X be a general curve of genus $g \geq 2$. Then $s(X) = 1$.*

If in addition X has very general moduli (so $[X]$ lies outside some countable collection of proper subvarieties of \mathcal{M}_g), then its Jacobian is simple and X neither dominates elliptic curves. Furthermore, the cardinality of the Iitaka-Severi set is still one for general plane curves of degree $d \geq 4$ and for general hyperplane sections of regular surfaces (cf. [10, 32]). It is worth noting that these facts may be proved either by a moduli count based on Hurwitz formula, or by a monodromy argument on Hodge structures.

In the light of the above mentioned results on curves, it is interesting to wonder whether—under some assumption of generality in a suitable moduli space—the same statement may hold true for higher dimensional varieties of general type.

Almost nothing general is known in this direction as the techniques used for curves hardly apply to higher dimensional varieties. On one hand, it is difficult to estimate properly the number of moduli of ramification divisors of dominant rational maps $X \dashrightarrow Y$. On the other, the Kodaira dimension is no longer governed by Hodge structures.

This work was partially supported by INdAM (GNSAGA); PRIN 2009 “Moduli, strutture geometriche e loro applicazioni”; FAR 2011 (PV) “Varietà algebriche, calcolo algebrico, grafi orientati e topologici”.

However, [16] establishes that general surfaces $X \subset \mathbb{P}^3$ of degree $5 \leq d \leq 11$ satisfy $s(X) = 1$, and the same is conjectured for general hypersurfaces $X \subset \mathbb{P}^n$ of general type and arbitrary degree.

Throughout we deal instead with this issue on products of curves, that is we investigate the existence of dominant rational maps $C \times D \dashrightarrow S$, where C and D are smooth complex projective curves and S is some surface of general type.

We recall that the product $C \times D$ is of general type if and only if both C and D are. Clearly, if either $s(C) > 1$ or $s(D) > 1$, then $s(C \times D) > 1$ as well. Moreover, there are several examples of surfaces of general type dominated by products of curves; beside Beauville surfaces (cf. [5, p.159] and [9, 2]), there are series of papers aiming to classify surfaces of general type dominated by products of curves under Galois rational coverings (see e.g. [3, 4, 30, 31]). In particular, the curves involved in these constructions always possess non-trivial automorphisms.

On the other hand, when both the factors are assumed to have very general moduli—hence without non-trivial automorphisms and maps on other curves of general type—we prove the following.

Theorem 1.2. *Let C and D be two distinct very general curves of genus $g \geq 7$ and $g' \geq 2$ respectively. Then $s(C \times D) = 1$.*

Hence the latter theorem leaves out only finitely many cases to discuss in view of a whole understanding of the problem.

This result is not a direct consequence of the one-dimensional case. In line with [16], we achieve Theorem 1.2 by using both the analogues of the techniques we mentioned above for curves. On one hand, we exploit a Hodge theoretical argument using monodromy to deduce that the possible target surfaces of dominant rational maps $C \times D \dashrightarrow S$ have null geometric genus. On the other, we perform a moduli count of products of curves, surfaces of general type having $p_g(S) = 0$, and families of curves lying on them. In particular, a very important role is played by the use of Mori's bend-and-break technique (cf. [26] and [23, Section II.5]) on isotrivial families of curves covering S , whereas [16] heavily uses Hurwitz formula and bounds on number of moduli of ramification divisors. Furthermore, in order to treat the case $g = 7$, we argue by degeneration to stable curves approaching the boundary of the moduli space $\mathcal{M}_{g'}$.

We point out that our argument fails to work for lower genera, but we believe possible some refinement leading to the assertion for g and g' satisfying $g + g' \geq 9$. Moreover, it is difficult to exhibit examples of dominant rational maps $C \times D \dashrightarrow S$ on surface of general type when $s(C) = s(D) = 1$ and the curves do not possess automorphisms. Therefore it seems natural to conjecture that Theorem 1.2 could be extended both to lower genera—except when $g = g' = 2$ because there always exists a dominant map $C \times D \dashrightarrow S$ (cf. Remark 4.2)—and to a Zariski open subset of $\mathcal{M}_g \times \mathcal{M}_{g'}$. Namely,

Conjecture 1.3. *Let C and D be two distinct general curves of genus $g \geq 2$ and $g' \geq 2$ respectively. Then*

$$s(C \times D) = \begin{cases} 2 & \text{if } g = g' = 2 \\ 1 & \text{otherwise} \end{cases}.$$

The plan of the paper is the following. In Section 2 we shall introduce the main preliminary results. In Section 3 we shall turn to isotrivial families of curves covering surfaces of general type. Moreover, we shall prove a rigidity result for dominant rational maps from products of curves to a fixed surface of general type. Finally, Section 4 shall be devoted to prove Theorem 1.2.

2. PRELIMINARIES

This section concerns the preliminary results involved in the proof of Theorem 1.2. After fixing notation, we shall recall some important facts and prove some results about the moduli space of stable curves and its boundary divisors. Then we shall follow [16, Section 2] to deal with Hilbert schemes and moduli of surfaces of general type. Finally, we shall turn to products of curves and their Hodge structures.

2.1. Notation. We work throughout over the field \mathbb{C} of complex numbers. By *variety* we mean a reduced algebraic variety over \mathbb{C} . When we speak of a *smooth* variety, we always implicitly assume it to be irreducible.

Given a variety X , we say that a property holds for a *general* point $x \in X$ if it holds on a Zariski open non-empty subset of X . Moreover, we say that $x \in X$ is a *very general* point if there exists a countable collection of proper subvarieties of X such that x is not contained in the union of those subvarieties.

As is customary, for any smooth surface X , we denote by $q(X) = \dim H^{1,0}(X)$ the *irregularity* and by $p_g(X) = \dim H^{2,0}(X)$ the *geometric genus*.

2.2. Moduli of curves. Let \mathcal{M}_g be the moduli space of smooth projective curves of genus $g \geq 2$, which is an irreducible variety of dimension $3g - 3$. Let $\overline{\mathcal{M}}_g$ denote the Deligne-Mumford compactification and let $\Delta = \overline{\mathcal{M}}_g - \mathcal{M}_g$ be the boundary divisor. We recall that the irreducible components Δ_i of Δ are closures of loci of curves with one node. In particular, the general curve of Δ_0 is irreducible, whereas for $i = 1, \dots, \lfloor g/2 \rfloor$, the general curve of Δ_i splits into two irreducible curves of genus i and $g - i$ attached at one point (see for instance [1, Chapter XII] and [20, Chapter 2]).

We are interested in closed subvarieties of $\overline{\mathcal{M}}_g$ intersecting Δ at some point representing curves with only rational and elliptic components. When $g = 2$, it is well known that the only complete subvarieties of \mathcal{M}_2 are points, hence any positive dimensional closed subvariety of $\overline{\mathcal{M}}_2$ parameterizes also curves having only rational and elliptic components.

Dealing with higher genera, the following holds (cf. [13, Corollary 2.2]) .

Theorem 2.1. *For $g = 3$ and $g \geq 5$, any two irreducible closed subvarieties of $\overline{\mathcal{M}}_g$ of codimension one have non-empty intersection.*

Therefore codimension-one subvarieties of \mathcal{M}_g satisfy the following property.

Corollary 2.2. *Let $g \geq 3$ and let $\mathcal{Z} \subset \mathcal{M}_g$ be an irreducible subvariety of codimension one. Then the closure $\overline{\mathcal{Z}} \subset \overline{\mathcal{M}}_g$ meets Δ_1 .*

Proof. Thanks to Theorem 2.1, the only case to treat is $g = 4$. Consider the Chow group $A^1(\overline{\mathcal{M}}_4)$ of codimension-one cycles of $\overline{\mathcal{M}}_4$, and let $\lambda, \delta_0, \delta_1, \delta_2$ be the standard independent divisor classes generating $A^1(\overline{\mathcal{M}}_4)$ as a vector space. Let $a\lambda + a_0\delta_0 + a_1\delta_1 + a_2\delta_2$ be the class of $\overline{\mathcal{Z}}$. If Δ_1 and $\overline{\mathcal{Z}}$ had empty intersection, the product of their classes would vanish in the Chow group $A^2(\overline{\mathcal{M}}_4)$ of codimension-two cycles. In particular, we would have a relation $(a\lambda + a_0\delta_0 + a_1\delta_1 + a_2\delta_2)\delta_1 = 0$ between the products of the standard classes, but this is impossible as the unique such a relation is $(10\lambda - \delta_0 - 2\delta_1)\delta_2 = 0$ (see [13, Section 2]). \square

Thus we deduce the following.

Proposition 2.3. *Let $g \geq 2$ and let $\mathcal{Z} \subset \mathcal{M}_g$ be an irreducible subvariety of codimension one. Then there exists $[Z'] \in \overline{\mathcal{Z}}$ such that Z' has only rational and elliptic components.*

Proof. The case $g = 2$ has been discussed above, so we set $g \geq 3$ and we proceed by induction on g . By Corollary 2.2, the closure $\overline{\mathcal{Z}} \subset \overline{\mathcal{M}}_g$ meets the boundary at Δ_1 . Hence there exists an irreducible complete subvariety $\Sigma \subset \overline{\mathcal{Z}} \cap \Delta_1$ of dimension $3g - 5$. In particular, if $[Z] \in \Sigma$, then Z consists of some curves Z_1 and Z_{g-1} attached at one point, with $[Z_1] \in \overline{\mathcal{M}}_1$ and $[Z_{g-1}] \in \overline{\mathcal{M}}_{g-1}$.

Suppose that for general $[Z] \in \Sigma$, the curve Z possesses a (unique) smooth component Z_{g-1} . Therefore the image of the projection $\pi: \Sigma \dashrightarrow \mathcal{M}_{g-1}$ has dimension $\dim \pi(\Sigma) \geq 3g - 7 = \dim \mathcal{M}_{g-1} - 1$. By induction, there exists $[Z'_{g-1}] \in \pi(\Sigma)$ such that Z'_{g-1} has only rational and elliptic components. As Σ is complete, there exists $[Z'] \in \Sigma \subset \overline{\mathcal{Z}}$, with Z' consisting of Z'_{g-1} and some elliptic curve Z'_1 attached at one point.

On the other hand, assume that the general $[Z] \in \Sigma$ has no smooth components of genus $g - 1$. Thus there exists a fixed $0 \leq i \leq \left\lfloor \frac{g-1}{2} \right\rfloor$ such that any $[Z]$ is obtained from some $[Z_1] \in \overline{\mathcal{M}}_1$ and $[Z_{g-1}] \in \Delta_i \subset \overline{\mathcal{M}}_{g-1}$. Viceversa, by irreducibility, completeness and $\dim \Sigma = \dim \overline{\mathcal{M}}_1 + \dim \Delta_i + 1$, we deduce that for any $[Z_1] \in \overline{\mathcal{M}}_1$ and $[Z_{g-1}] \in \Delta_i$, there exists $[Z] \in \Sigma$ given by $[Z_1]$ and $[Z_{g-1}]$. In particular, Δ_i parameterizes also curves having only rational and elliptic component, so that Σ does. \square

2.3. Families of curves. Let B be a smooth variety and let $\mathcal{E} \xrightarrow{q} B$ be a family of curves of genus g , that is a surjective proper morphism such that $E_b = q^{-1}(b)$ is a curve of genus g . We recall that if q is a smooth morphism, it is naturally defined a modular map $\mu: B \rightarrow \mathcal{M}_g$ as $\mu(b) = [E_b]$ (see e.g. [27, Chapter 5]). More generally, if q is not smooth and $\mathcal{E} = \bigcup \mathcal{E}_i$ is an irreducible decomposition such that any component dominates B , we can make a base change

$$\begin{array}{ccc} \mathcal{F}_i & \xrightarrow{\nu} & \mathcal{E}_i \\ p_i \downarrow & & \downarrow q \\ W_i & \longrightarrow & B \end{array}$$

such that $\mathcal{F}_i \xrightarrow{p_i} W_i$ is a smooth family of genus g curves and $\nu^{-1}(E_b) \rightarrow E_b$ is the normalization map (see [16, Section 2.3]). Thus we still define the modular map $\mu: W_i \rightarrow \mathcal{M}_g$ such that $\mu(w) = [F_w]$, and we define the *modular dimension* of the family $\mathcal{E} \xrightarrow{q} B$ as

$$M(\mathcal{E}/B) := \max_i \dim \mu(W_i).$$

Given a surface S of general type, let us denote by $\text{Hilb}(S)$ its Hilbert scheme. We assume further that $\mathcal{E} \xrightarrow{q} B$ is a family of curves on S , that is $\mathcal{E} \subset B \times S$. Over a Zariski open subset $U \subset B$, the morphism q is flat and we can define a map $\rho: U \rightarrow \text{Hilb}(S)$ sending $b \in U$ to the point parameterizing the curve E_b on S . Thus we define the *dimension of the family \mathcal{E} on S* as

$$D(\mathcal{E}/B) := \dim \rho(U),$$

and we have the following result based on Mori's bend-and-break (cf. [16, Proposition 2.3.2]).

Theorem 2.4. *Let $\mathcal{E} \rightarrow B$ be a family of curves of genus $g \geq 2$ on a surface of general type. Then*

$$M(\mathcal{E}/B) \leq D(\mathcal{E}/B) \leq M(\mathcal{E}/B) + 1.$$

Remark 2.5. We recall a well-known fact often involved in the proof of Theorem 1.2, that is neither rational nor elliptic curves cover a surface of general type. In particular, $D(\mathcal{E}/B) = 0$ for any family of curves on S of genus $g < 2$.

2.4. Moduli of surface of general type. Let $\mathcal{M}_{K^2, \chi}$ be the variety parameterizing isomorphism classes of surfaces having numerical invariants χ and K^2 . As in [8], we consider the isomorphism class $[S] \in \mathcal{M}_{K^2, \chi}$ of a surface of general type S , and we denote by \mathcal{M} the union of the components of $\mathcal{M}_{K^2, \chi}$ whose points are isomorphism classes of surfaces orientedly homeomorphic to $[S]$. Then the *number of moduli* $M(S)$ of S is defined as the dimension of \mathcal{M} at $[S]$, and it satisfies the following estimate (cf. [16, Theorem 2.5.1]).

Theorem 2.6. *Let S be a minimal surface of general type. Then*

$$M(S) \leq 11\chi(\mathcal{O}_S) + K_S^2.$$

2.5. Cohomology of products of curves. Let C and D be two curves of genus g and g' , respectively. Then the complex cohomology of the surface $C \times D$ is governed by *Künneth formula* (cf. [14, p.103-4]), that is

$$H^{p,q}(C \times D) \cong \bigoplus_{\substack{i+h=p \\ j+k=q}} H^{i,j}(C) \otimes H^{h,k}(D). \quad (2.1)$$

In particular, global canonical sections are such that $H^{2,0}(C \times D) \cong H^{1,0}(C) \otimes H^{1,0}(D)$. Hence the canonical map $\phi: C \times D \rightarrow \mathbb{P}^{gg'-1}$ factors as

$$C \times D \xrightarrow{\phi|_{\omega_C} \times \phi|_{\omega_D}} \mathbb{P}^{g-1} \times \mathbb{P}^{g'-1} \xrightarrow{\sigma} \mathbb{P}^{gg'-1}, \quad (2.2)$$

where $\phi|_{\omega_C}$ and $\phi|_{\omega_D}$ are the canonical maps of C and D , whereas σ is the Segre embedding (cf. [6, p.87]).

We note further that $H^{2,0}(C \times D)$ is isomorphic to the space of holomorphic two-forms of the Hodge substructure $H^1(C, \mathbb{C}) \otimes H^1(D, \mathbb{C})$, and we have the following.

Lemma 2.7. *Let C and D be two distinct very general curves of genus $g \geq 2$ and $g' \geq 2$, respectively. Then the Hodge structure $H^1(C, \mathbb{C}) \otimes H^1(D, \mathbb{C})$ is irreducible.*

Proof. Let us consider the Jacobian variety $J(C)$ of C and the cohomology group $H^1(J(C), \mathbb{C}) \cong H^1(C, \mathbb{C})$. As C is assumed to be a very general curve of genus g , the monodromy action on $H^1(C, \mathbb{C})$ of the symplectic group $\mathrm{Sp}(2g, \mathbb{Z})$ —of $2g \times 2g$ matrices preserving the intersection form on $J(C)$ —is irreducible. Analogously, we have that $\mathrm{Sp}(2g', \mathbb{Z})$ acts irreducibly on $H^1(D, \mathbb{C})$. Then the induced action of $\mathrm{Sp}(2g, \mathbb{Z}) \times \mathrm{Sp}(2g', \mathbb{Z})$ on $H^1(C, \mathbb{C}) \otimes H^1(D, \mathbb{C})$ is irreducible as well (see e.g. [28, Section I.2.7]). \square

3. ISOTRIVIAL FAMILIES OF CURVES AND RIGIDITY

In this section we deal with isotrivial families of curves dominating a surface S of general type. We are aimed at proving a rigidity result on dominant rational maps $C \times D \dashrightarrow S$ from a product of curves, when the first factor deforms in a family $\mathcal{C} \rightarrow T$ with $C_0 = C$.

We say that a family of curves $\mathcal{E} \xrightarrow{q} B$ is *isotrivial* if there exists a Zariski open subset $U \subset B$ such that the fibres $E_b = q^{-1}(b)$ with $b \in U$ are all isomorphic to a fixed smooth curve. Moreover,

such a family is called *trivial* when it is birational to the product $B \times D$ endowed with the projection $B \times D \xrightarrow{p} B$.

Now, let D be a smooth projective curve of genus $g \geq 2$, and let $\mathcal{E} \xrightarrow{q} B$ be a one-dimensional isotrivial family with general fibre $E_b \cong D$. Under this assumption, the family is dominated by a trivial family constructed with the following explicit base change (cf. [7, Section 2.4]). Let

$$B_0 := \{(b, \psi) \mid b \in B \text{ and } \psi: E_b \rightarrow D \text{ is an isomorphism}\} \quad (3.1)$$

and let $B' \subset B_0$ be any connected component dominating B . Then the fibred product

$$\begin{aligned} \mathcal{E}' &:= B' \times_B \mathcal{E} \\ &= \{(b, \psi, y) \mid b \in B, y \in E_b \text{ and } \psi: E_b \rightarrow D \text{ is an isomorphism}\} \end{aligned}$$

is isomorphic to $B' \times D$ under the map $(b, \psi, y) \mapsto ((b, \psi), \psi(y))$. Hence we have the rational map of families

$$\begin{array}{ccc} B' \times D & \xrightarrow{\beta} & \mathcal{E} \\ \downarrow & & \downarrow \\ B' & \dashrightarrow & B \end{array}$$

given by $((b, \psi), y) \mapsto \psi^{-1}(y)$. Furthermore, the fibred surface $B' \times D$ is somehow universal among trivial families dominating \mathcal{E} . Namely,

Lemma 3.1. *Let C and D be smooth projective curves of genus $g, g' \geq 2$. Let $\mathcal{E} \xrightarrow{q} B$ be a one-dimensional isotrivial family with general fibre isomorphic to D . For any dominant rational map of families*

$$\begin{array}{ccc} C \times D & \xrightarrow{\varphi} & \mathcal{E} \\ p \downarrow & & \downarrow q \\ C & \xrightarrow{h} & B \end{array}$$

there exist a curve B' and a rational map $h': C \dashrightarrow B'$ such that φ factors through $B' \times D$ as a map of families

$$\begin{array}{ccc} C \times D & \xrightarrow{h' \times Id_D} & B' \times D \\ & \searrow \varphi & \downarrow \beta \\ & & \mathcal{E} \end{array} .$$

Proof. Let $x \in C$ be a general point. The fibre $D_x = p^{-1}(x)$ coincides with the curve $\{x\} \times D$, which is a copy of D . Under this identification, the restriction $\varphi|_{\{x\} \times D}: D \rightarrow E_{h(x)}$ is an isomorphism. Therefore we may define a map

$$\begin{aligned} h': C &\dashrightarrow B_0 \\ x &\mapsto \left(h(x), (\varphi|_{\{x\} \times D})^{-1} \right) \end{aligned} \quad (3.2)$$

whose image is an irreducible component of B_0 . Then we define $B' \subset B_0$ to be a connected component containing the image $h'(C)$. Finally, for general $(x, y) \in C \times D$, we deduce

$$(\beta \circ (h' \times Id_D))(x, y) = \beta \left(\left(h(x), (\varphi|_{\{x\} \times D})^{-1} \right), y \right) = \varphi|_{\{x\} \times D}(y) = \varphi(x, y)$$

as claimed. \square

A family of dominant rational maps from products of curves to a surface S of general type may be viewed as a map $f: \mathcal{C} \times_T \mathcal{D} \dashrightarrow S$, where $\mathcal{C} \rightarrow T$ and $\mathcal{D} \rightarrow T$ are families of smooth curves, and the restrictions $f_t: C_t \times D_t \dashrightarrow S$ are dominant.

When one of the factors does not move, i.e. $\mathcal{D} = T \times D$ for some smooth curve D , we have that $\mathcal{C} \times_T \mathcal{D} \cong \mathcal{C} \times D$. In this setting, we combine Theorem 2.4 with Lemma 3.1, and we prove that the maps of the family $f: \mathcal{C} \times D \dashrightarrow S$ factor through some fixed surface $B' \times D$. Namely,

Proposition 3.2. *Let D be a smooth projective curve of genus $g' \geq 2$ and let S be a surface of general type. Let $\mathcal{C} \rightarrow T$ be a family of smooth projective curves of genus $g \geq 2$ and let $f: \mathcal{C} \times D \dashrightarrow S$ be a rational map such that the restrictions $f_t: C_t \times D \dashrightarrow S$ are dominant. Moreover, assume that for general $x \in C_t$, the curve $\{x\} \times D$ maps birationally onto its image $f_t(\{x\} \times D) \subset S$.*

Then there exist a curve B' , and for general $t \in T$, a rational map $h'_t: C_t \dashrightarrow B'$ such that f_t factors through $B' \times D$ as

$$\begin{array}{ccc} C_t \times D & \xrightarrow{h'_t \times \text{Id}_D} & B' \times D \\ & \searrow f_t & \downarrow \\ & & S \end{array} \quad (3.3)$$

Proof. Let $t \in T$ be a general point. Our aim is to construct an isotrivial family $\mathcal{E} \xrightarrow{q} B$ as in Lemma 3.1, being independent of $t \in T$ and fitting in a commutative diagram as

$$\begin{array}{ccc} C_t \times D & \xrightarrow{\varphi_t} & \mathcal{E} \\ p_t \downarrow & & \downarrow q \\ C_t & \xrightarrow{h_t} & B. \end{array} \quad (3.4)$$

We recall that the fibres $p_t^{-1}(x) = \{x\} \times D$ form a family of copies of D with total space $C_t \times D$. As f_t is dominant, the images $f_t(\{x\} \times D)$ produce a one-dimensional family of curves covering S , which is parameterized over some curve $B_t \subset \text{Hilb}(S)$.

Let $B := (\bigcup_{t \in T} B_t)_{\text{red}} \subset \text{Hilb}(S)$ and let $\mathcal{F} \subset B \times S$ be the restriction to B of the universal family over $\text{Hilb}(S)$. We want to prove that B is a curve. By assumption, for general $t \in T$ and $x \in C_t$, the curve $\{x\} \times D$ is birational onto its image under f_t , and hence the normalization of $f_t(\{x\} \times D)$ is isomorphic to D . Therefore the modular dimension of $\mathcal{F} \rightarrow B$ is $M(\mathcal{F}/B) = 0$. Furthermore, the curve B_t does not depend on $t \in T$. Indeed, if B_t were deformed on the Hilbert scheme of S as we vary $t \in T$, the dimension of $\mathcal{F} \rightarrow B$ on S would be $D(\mathcal{F}/B) = \dim B \geq 2$, but this contradicts Theorem 2.4. Thus $\mathcal{F} \rightarrow B$ is a one-dimensional family of curves on S , whose general fibre F_b has normalization D .

Let $h_t: C_t \rightarrow B$ be the map sending $x \in C_t$ to the point $b \in B$ such that $f_t(\{x\} \times D) = F_b$. Then we define the dominant rational map

$$\begin{aligned} \psi_t: C_t \times D & \dashrightarrow \mathcal{F} \\ (x, y) & \longmapsto (h_t(x), f_t(x, y)). \end{aligned}$$

Normalizing \mathcal{F} —and possibly shrinking B —we obtain an isotrivial family of curves $\tilde{\mathcal{F}} \xrightarrow{q} B$ whose general fibre \tilde{F}_b is isomorphic to D . Furthermore, we can lift $\psi_t: C_t \times D \dashrightarrow \mathcal{F}$ to a dominant rational map of families $\varphi_t: C_t \times D \dashrightarrow \tilde{\mathcal{F}}$ fitting in (3.4). Thus Lemma 3.1 assures that there exist

a rational map $h'_t: C_t \dashrightarrow B_0$ as in (3.2) and a connected component $B' \subset B_0$ containing $h'_t(C_t)$, such that

$$\begin{array}{ccc} C_t \times D & \xrightarrow{\quad h'_t \times \text{Id}_D \quad} & B' \times D \\ & \searrow \varphi_t & \downarrow \beta \\ & & \widetilde{\mathcal{F}} \end{array} .$$

Clearly, any $h'_t: C_t \dashrightarrow B_0$ maps on some irreducible component of B_0 . Over an open subset of T , we can then assume that the image $h'_t(C_t)$ is independent of t , so that B' also is.

Finally, let $\pi: \widetilde{\mathcal{F}} \rightarrow S$ be the map inherited from the natural projection of $\mathcal{F} \subset B \times S$. By construction we have $f_t = \pi \circ \varphi_t$ as rational maps, and hence the following commutative diagram

$$\begin{array}{ccc} C_t \times D & \xrightarrow{\quad h'_t \times \text{Id}_D \quad} & B' \times D \\ & \searrow f_t & \downarrow \pi \circ \beta \\ & & S \end{array} .$$

□

Remark 3.3. Under the hypothesis of the proposition, we assume further that for general $t \in T$ and $y \in D$, the curve $C_t \times \{y\}$ maps birationally onto its image $f_t(C_t \times \{y\}) \subset S$. Then the general C_t is birational to an irreducible component of B' , and hence the family $\mathcal{C} \rightarrow T$ is isotrivial. In particular, any map $C \times D \dashrightarrow S$ is rigid under deformations as above of the first factor.

Remark 3.4. More generally, it would be very interesting to have a rigidity theorem for dominant rational maps $C \times D \dashrightarrow S$, which deform in families $f: \mathcal{C} \times_T \mathcal{D} \dashrightarrow S$.

For instance, by using the techniques of this paper, it is possible to prove that if D has genus $g' = 2$, it cannot move and the maps $f_t: C_t \times D \dashrightarrow S$ fit in (3.3).

In particular, if both C and D have genus 2, then $C \times D \dashrightarrow S$ is rigid as a map from products of smooth curves to a fixed surface of general type.

4. PROOF OF THEOREM 1.2

This section is devoted to prove Theorem 1.2. We start with a preliminary lemma providing restrictions on surfaces of general type dominated by products of very general curves. Namely,

Lemma 4.1. *Let C and D be two distinct very general curves of genus $g > 2$ and $g' \geq 2$, respectively. Let S be a minimal surface of general type and let $f: C \times D \dashrightarrow S$ be a dominant rational map of degree $m > 1$. Then*

- (i) $p_g(S) = 0$,
- (ii) $M(S) \leq 19$.

Proof. (i) Let

$$\begin{array}{ccc} X & & \\ \downarrow h & \searrow \tilde{f} & \\ C \times D & \xrightarrow{\quad f \quad} & S \end{array}$$

be a resolution of the indeterminacy locus of f , and let $f^*: H^2(S, \mathbb{C}) \rightarrow H^2(C \times D, \mathbb{C})$ be the Hodge structure map defined as the composition of the pullback map $\tilde{f}^*: H^2(S, \mathbb{C}) \rightarrow H^2(X, \mathbb{C})$ with the Gysin map $h_*: H^2(X, \mathbb{C}) \rightarrow H^2(C \times D, \mathbb{C})$.

Let us consider the injective morphism $f_{2,0}^*: H^{2,0}(S) \rightarrow H^{2,0}(C \times D)$, and let us recall that $H^{2,0}(C \times D) \cong H^{1,0}(C) \otimes H^{1,0}(D)$ is the holomorphic part in the Hodge decomposition of $H^1(C, \mathbb{C}) \otimes H^1(D, \mathbb{C})$. Then Lemma 2.7 assures that the image of the monomorphism $f_{2,0}^*$ is either trivial or the whole $H^{2,0}(C \times D)$. Thus either $p_g(S) = 0$ or $H^{2,0}(S) \cong H^{2,0}(C \times D)$.

Aiming for a contradiction, let us assume $H^{2,0}(S) \cong H^{2,0}(C \times D)$. Therefore the canonical map ϕ of $C \times D$ factors—as a rational map—through f and we have the following diagram

$$\begin{array}{ccc} C \times D & \xrightarrow{\phi} & \mathbb{P}^{gg'-1} \\ & \searrow f \quad \nearrow & \\ & S & \end{array} \quad (4.1)$$

If $g' > 2$, both C and D are embedded by their canonical maps, and hence ϕ is an embedding by (2.2). Thus we have a contradiction as $\deg f > 1$.

On the other hand, suppose that $g' = 2$. Then the canonical map of D factors through the hyperelliptic map. Hence $\deg \phi = 2$ and the canonical image $\phi(C \times D)$ is birational to $C \times \mathbb{P}^1$. Thus $\deg f = 2$ and S is birational to $C \times \mathbb{P}^1$ as well, which is still a contradiction as S is of general type.

(ii) Since S a minimal surface of general type, we have $\chi(\mathcal{O}_S) = 1 - q(S) + p_g(S) \geq 1$, and hence $\chi(\mathcal{O}_S) = 1$ by the first part of the proof. By Theorem 2.6 we have $M(S) \leq 11\chi(\mathcal{O}_S) + K_S^2$ and Miyaoka-Bogomolov inequality assures that $K_S^2 \leq 9\chi(\mathcal{O}_S)$. Furthermore, if $K_S^2 = 9$, then S is rigid by Yau's theorem. Thus the number of moduli of S satisfies $M(S) \leq 19$. \square

Remark 4.2. Let C and D be distinct very general curves of genus 2, with hyperelliptic involution i and j , respectively. Then their product admits a dominant map $C \times D \dashrightarrow Y$ on a surface of general type having $q(Y) = 0$ and $p_g(Y) = 4$, where Y is the quotient of $C \times D$ under the involution $(p, q) \mapsto (i(p), j(q))$. Furthermore, Y is the unique surface of general type dominated by $C \times D$ having positive geometric genus.

Indeed, if $f: C \times D \dashrightarrow S$ were a dominant rational map on another surface of general type with $p_g(S) > 0$, we would argue as in Lemma 4.1 and f would fit in (4.1). Since the monodromy group $M(\phi)$ of the canonical map is isomorphic to $\langle i, j \rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, the surfaces with positive geometric genus fitting in the diagram are just Y , $\mathbb{P}^1 \times D$ and $C \times \mathbb{P}^1$.

Moreover, we recall that a very general curve C of genus $g \geq 2$ does not dominate other curves of positive genus. So if $f: C \rightarrow E$ is a non-constant morphism to a curve E with normalization \tilde{E} , then either $\tilde{E} \cong C$ —and f is birational—or $\tilde{E} \cong \mathbb{P}^1$.

We now prove our result. Firstly, we shall use rigidity of dominant rational maps to set a moduli count excluding the case $g \geq 8$. In order to rule out the case of genus 7, we shall study the degeneration of trivial families of curves of genus g' dominating a fixed surface S of general type. Arguing by contradiction and using Proposition 2.3, we shall obtain some family of curves having only rational and elliptic components, which still covers S .

Proof of Theorem 1.2. Let C and D be two distinct very general curves of genus $g \geq 7$ and $g' \geq 2$ respectively. Without loss of generality, we set $g \geq g'$. By contradiction, let us assume the existence

of a dominant rational map $C \times D \dashrightarrow S$ of degree $m > 1$ on a surface S of general type and—up to consider the minimal model of S —let us suppose S to be smooth and minimal.

We define the locus $\mathcal{S} \subset \mathcal{M}_g \times \mathcal{M}_{g'}$ as

$$\mathcal{S} := \{([X], [Y]) \in \mathcal{M}_g \times \mathcal{M}_{g'} \mid \exists X \times Y \dashrightarrow S \text{ dominant}\}, \quad (4.2)$$

and let $\mathcal{R} \subset \mathcal{S}$ be an irreducible component passing through $([C], [D])$, endowed with the projection maps $\pi_1: \mathcal{R} \rightarrow \mathcal{M}_g$ and $\pi_2: \mathcal{R} \rightarrow \mathcal{M}_{g'}$.

Claim 4.3. *The projection map $\pi_2: \mathcal{R} \rightarrow \mathcal{M}_{g'}$ is generically finite.*

Proof. We consider the fibre over the very general point $[D] \in \mathcal{M}_{g'}$,

$$\pi_2^{-1}([D]) = \{([X], [D]) \in \mathcal{R} \mid \exists X \times D \dashrightarrow S \text{ dominant}\}.$$

Aiming for a contradiction, we assume $\dim \pi_2^{-1}([D]) > 0$. Let $\mathcal{T} := \pi_1(\pi_2^{-1}([D])) \subset \mathcal{M}_g$ and notice that it has the same dimension of $\pi_2^{-1}([D])$.

Since $[C] \in \mathcal{T}$ is a very general point of \mathcal{M}_g , it lies on the locus \mathcal{M}_g^0 of curves without automorphisms other than the identity. Let $T := \mathcal{T}|_{\mathcal{M}_g^0}$ and consider the restriction $\mathcal{C} \rightarrow T$ of the universal family over \mathcal{M}_g^0 . By construction, any fibre C_t is a curve admitting a dominant rational map $f_t: C_t \times D \dashrightarrow S$. Up to make a base change, the family of curves $\mathcal{C} \rightarrow T$ is endowed with a dominant rational map $f: \mathcal{C} \times D \dashrightarrow S$ having restrictions $f_t: C_t \times D \dashrightarrow S$.

We note further that for general $t \in T$ and $x \in C_t$, the curve $\{x\} \times D$ maps birationally onto its image $f_t(\{x\} \times D)$. Indeed $\{x\} \times D$ is a copy of the very general curve D , therefore $f_t(\{x\} \times D)$ is either birational to D or a rational curve. In the latter case, the surface S would be covered by rational curves, but this is impossible (cf. Remark 2.5).

Therefore the family $\mathcal{C} \rightarrow T$ fulfils the hypothesis of Proposition 3.2. Thus there exist a curve B' and—for general $t \in T$ —a non-constant map $h'_t: C_t \dashrightarrow B'$ such that

$$\begin{array}{ccc} C_t \times D & \xrightarrow{h'_t \times \text{Id}_D} & B' \times D \\ & \searrow f_t & \downarrow \\ & & S \end{array} \quad (4.3)$$

In particular, there is an irreducible component of B' dominated by the general fibre C_t , and hence by the very general curve C . Therefore the closure of such a component must be a rational curve. Then the surface S is covered by a family of rational curves under the map $B' \times D \dashrightarrow S$ in (4.3). Thus we have a contradiction as S is of general type. \square

Thanks to Lemma 4.1 we have that $p_g(S) = 0$ and the modular dimension of S satisfies $M(S) \leq 19$. Furthermore, for a general choice of S among minimal surfaces of general type having $p_g = 0$ and being dominated by a product of very general curves, we have

$$\dim \mathcal{R} \geq \dim(\mathcal{M}_g \times \mathcal{M}_{g'}) - M(S) \geq (3g - 3) + (3g' - 3) - 19. \quad (4.4)$$

Moreover, the generic finiteness of $\pi_2: \mathcal{R} \rightarrow \mathcal{M}_{g'}$ assures that $\dim \mathcal{R} \leq \dim \mathcal{M}_{g'} = 3g' - 3$. Thus $(3g - 3) - 19 \leq 0$ and the assertion of Theorem 1.2 is proved for any $g \geq 8$ and $g' \geq 2$.

Then we assume $g = 7$ and $2 \leq g' \leq 7$. As a consequence of inequality (4.4) and Claim 4.3 we deduce that the image $\mathcal{Z} := \pi_2(\mathcal{R}) \subset \mathcal{M}_{g'}$ is a subvariety of dimension $3g' - 4 \leq \dim \mathcal{Z} \leq 3g' - 3$.

Let $\overline{\mathcal{Z}} \subset \overline{\mathcal{M}}_{g'}$ be its closure. Therefore Proposition 2.3 assures that there exists $[Z'] \in \overline{\mathcal{Z}}$ such that all the irreducible components of Z' are rational and elliptic.

Claim 4.4. *For any $[Z] \in \overline{\mathcal{Z}} - \mathcal{Z}$, there exist a family $\mathcal{X} \rightarrow W$ of nodal curves of genus g' and a dominant rational map $\mathcal{X} \dashrightarrow S$, such that the general fibre X_w of the family consists of a curve birational to Z and some rational components.*

Proof. Let Z be a curve of genus g' such that $[Z] \in \overline{\mathcal{Z}} - \mathcal{Z}$. Let U be a disk parameterizing a family $\mathcal{Y} \rightarrow U$ of curves such that $[Y_t] \in \mathcal{Z}$ for $t \neq 0$ and $Y_0 = Z$. By construction for any $t \neq 0$, there exists a dominant rational map $f_t: X_t \times Y_t \dashrightarrow S$, with $[X_t] \in \mathcal{M}_g$. In particular, the images of the curves $\{x\} \times Y_t$ under the maps f_t describe a family of curves covering S .

We focus on those curves $f_t(\{x\} \times Y_t) \subset S$ passing through a fixed general point $s \in S$. Up to some base change, we can assume the maps f_t varying holomorphically on $U^* = U - \{0\}$. Hence we can define a dominant rational map

$$\begin{array}{c} \mathcal{Y} \xrightarrow{\xi} S \\ \downarrow \\ U^* \end{array},$$

where the restrictions $\xi|_{Y_t}$ are inherited from the maps f_t , and the curves $\xi(Y_t)$ pass through $s \in S$. We extend ξ at $t = 0$ by nodal reduction (cf. [20, p.119])

$$\begin{array}{c} \mathcal{Y}' \xrightarrow{\xi'} S \\ \downarrow \\ U' \end{array},$$

so that the central fibre Y'_0 is a nodal curve containing a curve birational to Z and the remaining components are rational. Moreover, the image $\xi'(Y'_0) \subset S$ is a curve passing through the general point $s \in S$, and hence the assertion follows. \square

Since $[Z'] \in \overline{\mathcal{Z}} - \mathcal{Z}$, there exists a family $\mathcal{X}' \rightarrow W$ and a dominant rational map $\mathcal{X}' \dashrightarrow S$ as in Claim 4.4. In particular, each component of the general fibre X'_w is either rational or elliptic. Hence the surface of general type S is covered by curves of genus smaller than two. Thus we get a contradiction and Theorem 1.2 is proved. \square

Remark 4.5. We note that when $2 \leq g' \leq g \leq 6$, the subvariety $\mathcal{Z} \subset \mathcal{M}_{g'}$ has no longer $\text{codim} \mathcal{Z} \leq 1$, and our argument fails to work. On the other hand, we are assured that $\overline{\mathcal{Z}}$ meets the boundary $\Delta \subset \overline{\mathcal{M}}_{g'}$ when $\text{codim} \mathcal{Z} \leq 2g' - 2$ (cf. [12]). Hence one can hope to extend Theorem 1.2 to lower genera by refining our techniques and using some results describing the intersection between $\overline{\mathcal{Z}}$ and some $\Delta_i \subset \overline{\mathcal{M}}_{g'}$ (see e.g. [11, 24] for Δ_0).

ACKNOWLEDGEMENTS

We would like to thank Alessandro Ghigi and Valeria Marucci for helpful suggestions.

REFERENCES

- [1] E. Arbarello, M. Cornalba, P. A. Griffiths, *Geometry of algebraic curves*, Volume II, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences] **268**, Springer, Heidelberg, 2011.
- [2] I. Bauer, F. Catanese, Some new surfaces with $p_g = q = 0$, in *The Fano Conference*, 123–142, Univ. Torino, Turin, 2004.
- [3] I. Bauer, F. Catanese, F. Grunewald, The classification of surfaces with $p_g = q = 0$ isogenous to a product of curves, *Pure Appl. Math. Q.* **4** (2008), 547–586.
- [4] I. Bauer, R. Pignatelli, The classification of minimal product-quotient surfaces with $p_g = 0$, to appear in *Math. Comp.*.
- [5] A. Beauville, *Surfaces algébriques complexes*, Astérisque **54**, Soc. Math. France, Paris, 1978.
- [6] A. Beauville, *Complex algebraic surfaces*, Second edition, London Mathematical Society Student Texts **34**, Cambridge University Press, Cambridge, 1996.
- [7] L. Caporaso, J. Harris, B. Mazur, Uniformity of rational points, *J. Amer. Math. Soc.* **10** (1997), 1–35.
- [8] F. Catanese, On the moduli spaces of surfaces of general type, *J. Differential Geom.* **19** (1984), 483–515.
- [9] F. Catanese, Fibred surfaces, varieties isogenous to a product and related moduli spaces, *Amer. J. Math.* **122** (2000), 1–44.
- [10] C. Ciliberto, G. van der Geer, On the Jacobian of a hyperplane section of a surface, in *Classification of irregular varieties (Trento, 1990)*, 33–40, Lecture Notes in Math. **1515**, Springer, Berlin, 1992.
- [11] E. Colombo, G. P. Pirola, Some density results for curves with nonsimple Jacobians, *Math. Ann.* **288** (1990), 161–178.
- [12] S. Diaz, A bound on the dimensions of complete subvarieties of \mathcal{M}_g , *Duke Math. J.* **51** (1984), 405–408.
- [13] C. F. Faber, Some results on the codimension-two Chow group of the moduli space of stable curves, in *Algebraic curves and projective geometry (Trento, 1988)*, 66–75, Lecture Notes in Math. **1389**, Springer, Berlin, 1989.
- [14] P. A. Griffiths, J. Harris, *Principles of Algebraic Geometry*, Pure and Applied Mathematics, Wiley Interscience, New York, 1978.
- [15] L. Guerra, Complexity of Chow varieties and number of morphisms on surfaces of general type, *Manuscripta Math.* **98** (1999), 1–8.
- [16] L. Guerra, G. P. Pirola, On rational maps from a general surface in \mathbb{P}^3 to surfaces of general type, *Adv. Geom.* **8** (2008), 289–307.
- [17] L. Guerra, G. P. Pirola, On the finiteness theorem for rational maps on a variety of general type, *Collect. Math.* **60** (2009), 261–276.
- [18] L. Guerra, G. P. Pirola, Parametrization of rational maps on a variety of general type, and the finiteness theorem, to appear in *Proc. Amer. Math. Soc.*.
- [19] C. D. Hacon, J. McKernan, Boundedness of pluricanonical maps of varieties of general type, *Invent. Math.* **166** (2006), 1–25.
- [20] J. Harris, I. Morrison, *Moduli of curves*, Graduate Texts in Mathematics **187**, Springer-Verlag, New York, 1998.
- [21] G. Heier, Effective finiteness theorems for maps between canonically polarized compact complex manifolds, *Math. Nachr.* **278** (2005), 133–140.
- [22] E. Kani, Bounds on the number of nonrational subfields of a function field, *Invent. Math.* **85** (1986), 185–198.
- [23] J. Kollár, Rational curves on algebraic varieties, *Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]* **32**, Springer-Verlag, Berlin, 1996.
- [24] I. Krichever, Real normalized differentials and compact cycles in the moduli space of curves, preprint (2012), arXiv:1204.2192v1.
- [25] K. Maehara, A finiteness property of varieties of general type, *Math. Ann.* **262** (1983), 101–123.
- [26] S. Mori, Projective manifolds with ample tangent bundles, *Ann. of Math.* **110** (1979), 593–606.
- [27] D. Mumford, J. Fogarty, *Geometric invariant theory*, Second edition, *Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas]* **34**, Springer-Verlag, Berlin, 1982.
- [28] M. A. Naïmark, A. I. Štern, *Theory of group representations*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences] **246**, Springer-Verlag, New York, 1982.

- [29] J. C. Naranjo, G. P. Pirola, Bounds of the number of rational maps between varieties of general type, *Amer. J. Math.* **129** (2007), 1689–1709.
- [30] M. Penegini, The classification of isotrivially fibred surfaces with $p_g = q = 2$, *Collect. Math.* **62** (2011), 239–274.
- [31] F. Polizzi, Standard isotrivial fibrations with $p_g = q = 1$, *J. Algebra* **321** (2009), 1600–1631.
- [32] F. Severi, Le corrispondenze fra i punti di una curva variabile in un sistema lineare sopra una superficie algebrica, *Math. Ann.* **74** (1913), 515–544.
- [33] S. Takayama, Pluricanonical systems on algebraic varieties of general type, *Invent. Math.* **165** (2006), 551–587.
- [34] M. Tanabe, Bounds on the number of holomorphic maps of compact Riemann surfaces, *Proc. Amer. Math. Soc.* **133** (2005), 3057–3064.
- [35] H. Tsuji, Pluricanonical systems of projective varieties of general type II, *Osaka J. Math.* **44** (2007), 723–764.

DIPARTIMENTO DI MATEMATICA E APPLICAZIONI, UNIVERSITÀ DEGLI STUDI DI MILANO-BICOCCA, VIA COZZI 53, 20125 MILANO - ITALY

E-mail address: `francesco.bastianelli@unimib.it`

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DEGLI STUDI DI PAVIA, VIA FERRATA 1, 27100 PAVIA - ITALY

E-mail address: `gianpietro.pirola@unipv.it`